

MTH250 - CALCULUS

Lecture No. 20

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LECTURE 20

Integration IV

In this lecture, we concentrate on techniques for evaluating trigonometric integrals, substitutions involving trigonometric functions and the technique of partial fractions.

20.1 Trigonometric integrals

Example 20.1. Find $\int \sec x dx$.

Solution. We multiply numerator and denominator by $\sec x + \tan x$. Then

$$\int \sec x dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) = \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \right) dx.$$

If we substitute $u = \sec x + \tan x$, then $du = (\sec x \tan x + \sec^2 x) dx$. The integral becomes

$$\int \left(\frac{1}{u} \right) du = \ln |u| + C.$$

Thus, we have

$$\int \sec x dx = \ln |\sec x + \tan x| + C.$$

□

Example 20.2. Find $\int \cos^3 x dx$.

Solution. Simply substituting $u = \cos x$ isn't helpful, since then

$$du = -\sin x dx.$$

In order to integrate powers of cosine, we would need an extra $\sin x$ factor. Similarly, a power of sine would require an extra $\cos x$ factor. We can separate one cosine factor and convert the remaining $\cos^2 x$ factor to an expression involving sine using the identity

$$\sin^2 x + \cos^2 x = 1.$$

We can then evaluate the integral by substituting $u = \sin x$.

$$\begin{aligned} \int \cos^3 x dx &= \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx \\ &= \int (1 - u^2) du = u - \frac{1}{3} u^3 + C = \sin x - \frac{1}{3} \sin^3 x + C. \end{aligned}$$

□

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor. The remainder of the expression can be in terms of cosine. We could also try only one cosine factor. The remainder of the expression can be in terms of sine. The identity $\cos^2 x + \sin^2 x$ enables us to convert back and forth between even powers of sine and cosine.

Example 20.3. Evaluate $\int \sin^5 x \cos^2 x dx$.

Solution. Substituting $u = \cos x$, we have $du = -\sin x dx$. So,

$$\begin{aligned} \int \sin^5 x \cos^2 x dx &= \int (\sin^2 x)^2 \cos^2 x dx, \\ &= \int (1 - \cos^2 x)^2 \cos^2 x dx = \int (1 - u^2)^2 u^2 (-du) \\ &= - \int (u^2 - 2u^4 + u^6) du = - \left(\frac{u^3}{3} - 2 \frac{u^5}{5} + \frac{u^7}{7} \right) - C \\ &= -\frac{1}{2} \cos^3 x + \frac{2}{3} \cos^5 x - \frac{1}{7} \cos^7 x - C. \end{aligned}$$

□

Remark 20.1. If the integrand contains even powers of both sine and cosine, this strategy fails. In that case, we can take advantage of the following half-angle identities

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2} (1 + \cos 2x).$$

Example 20.4. Find $\int_0^\pi \sin^2 x dx$.

Solution. Using the half-angle formula for $\sin^2 x$, we have

$$\begin{aligned}\int_0^\pi \sin^2 x dx &= \frac{1}{2} \int_0^\pi (1 - \cos 2x) dx = \left[\frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^\pi \\ &= \frac{1}{2} \left(\pi - \frac{1}{2} \sin 2\pi \right) - \frac{1}{2} \left(0 - \frac{1}{2} \sin 0 \right) = \frac{1}{2} \pi.\end{aligned}$$

□

20.1.1 Guidelines for trigonometric integrals

When evaluating integrals of the form

$$\int \sin^m x \cos^n x dx, \quad m, n \geq 0$$

- If the power of sine is odd ($m = 2k + 1$), save one sine factor.
- Use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine

$$\int \sin^{2k+1} x \cos^n x dx = \int (\sin^2)^k \cos^n x \sin x dx = \int (1 - \cos^2)^k \cos^n x \sin x dx.$$

- Then, substitute $u = \cos x$.
- If the powers of both sine and cosine are even, use the half-angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

- Sometimes, it is helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x.$$

- This also works for integrals of the form

$$\int \tan^m x \sec^n x dx \quad \text{and} \quad \int \cot^m x \csc^n x dx$$

Example 20.5. Evaluate $\int \tan^6 x \sec^4 x dx$.

Solution. We use the identity $\sec^2 x = 1 + \tan^2 x$. Then, substitute $u = \tan x$ so that $du = \sec^2 x$.

$$\begin{aligned}\int \tan^6 x \sec^4 x dx &= \int \tan^6 x \sec^2 x \sec^2 x dx \\ &= \int \tan^6 x (1 + \tan^2 x) \sec^2 x dx \\ &= \int u^6 (1 + u^2) du = \int (u^6 + u^8) du \\ &= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C.\end{aligned}$$

□

Example 20.6. Evaluate $\int \sin 4x \cos 5x dx$.

Solution. We use the identity

$$2 \sin A \cos B = \sin(A - B) + \sin(A + B)$$

with $A = 4x$ and $B = 5x$. Therefore,

$$\begin{aligned} \int \sin 4x \cos 5x dx &= \frac{1}{2} \int [\sin(-x) + \sin 9x] dx = \frac{1}{2} \int (-\sin x + \sin 9x) dx \\ &= \frac{1}{2} (\cos x - \frac{1}{9} \cos 9x) + C. \end{aligned}$$

□

In general: we use the following identities to evaluate special types of trigonometric integrals.

Integral	Identity
$\int \sin Ax \cos Bx dx$	$2 \sin A \cos B = \sin(A - B) + \sin(A + B)$
$\int \sin Ax \sin Bx dx$	$2 \sin A \cos B = \cos(A - B) - \cos(A + B)$
$\int \cos Ax \cos Bx dx$	$2 \sin A \cos B = \cos(A - B) + \cos(A + B)$

20.2 Trigonometric substitutions

Sometimes, it is helpful to use following trigonometric substitution. In each case, the restriction on θ is imposed to ensure that the function that defines the substitution is one-to-one.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta.$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta.$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta \leq \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta.$

Example 20.7. Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2} dx$.

Solution. Let $x = 3 \sin \theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then, $dx = 3 \cos \theta d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta} = \sqrt{9 \cos^2 \theta} = 3|\cos \theta| = 3 \cos \theta.$$

Therefore,

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2} dx &= \frac{3 \cos \theta}{9 \sin^2 \theta} 3 \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\ &= \int \cot^2 \theta = \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C.\end{aligned}$$

□

As this is an indefinite integral, we must return to the original variable x . As $\sin \theta = \frac{x}{3}$ therefore, $\theta = \sin^{-1} \frac{x}{3}$. Moreover,

$$\sin^2 \theta = \frac{x^2}{9} \implies \cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{x^2}{9}.$$

Therefore,

$$\cos \theta = \frac{\sqrt{9-x^2}}{3} \quad \text{and} \quad \cot \theta = \frac{\sqrt{9-x^2}}{x}.$$

Hence,

$$\int \frac{\sqrt{9-x^2}}{x^2} dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \left(\frac{x}{3} \right) + C.$$

Example 20.8. Evaluate $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx$.

Solution. We use the substitution $x = \frac{3}{2} \tan \theta$, then

$$dx = \frac{3}{2} \sec^2 \theta \quad \text{and} \quad \sqrt{4x^2+9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sec \theta.$$

$$- \quad x = 0 \implies \tan \theta = 0 \implies \theta = 0.$$

$$- \quad x = \frac{3\sqrt{3}}{2} \implies \tan \theta = \sqrt{3} \implies \theta = \frac{\pi}{3}.$$

$$\begin{aligned}\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx &= \int_0^{\pi/3} \frac{\frac{27}{8} \tan^3 \theta}{27 \sec^3 \theta} \frac{3}{2} \sec^2 \theta d\theta = \int_0^{\pi/3} \frac{3 \tan^3 \theta}{16 \sec \theta} d\theta, \\ &= \frac{3}{16} \int_0^{\pi/3} \frac{\sin^3 \theta}{\cos^2 \theta} d\theta = \int_0^{\pi/3} \frac{3}{16} \frac{1 - \cos^2 \theta}{\cos^2 \theta} \sin \theta d\theta\end{aligned}$$

Now, we use another substitution $u = \cos \theta$, then $du = -\sin \theta d\theta$.

$$- \quad \theta = 0 \implies u = 1.$$

$$- \quad \theta = \frac{\pi}{3} \implies u = \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx &= -\frac{3}{16} \int_1^{1/2} \frac{1-u^2}{u^2} du = -\frac{3}{16} \int_1^{1/2} (1-u^{-2}) du \\ &= \frac{3}{16} \left[u + \frac{1}{u} \right]_1^{1/2} = \frac{3}{16} \left[\left(\frac{1}{2} + 2 \right) - (1+1) \right] = \frac{3}{32}.\end{aligned}$$

□

20.3 Partial fractions technique for integration

Recall that a fraction is termed *proper* when the degree of the numerator is less than the degree of the denominator. It is termed *improper* otherwise. For example, $\frac{x+1}{x^2+2}$

is proper and $\frac{x^2}{(x+1)(x+2)}$ is improper fraction.

The improper fractions can be simplified using algebraic division, e.g

$$\frac{x^3+4x^2-x+2}{x^2+x} = x+3 + \frac{2-4x}{x^2+x}$$

20.3.1 Partial fraction

In order to simplify an expression of the form $\frac{3}{x+2} - \frac{5}{x-5}$, we take the common denominator which is $(x+2)(x-5)$ and proceed as follows:

$$\frac{3}{x+2} - \frac{5}{x-5} = \frac{3(x-5) - 5(x+2)}{(x+2)(x-5)} = \frac{3x-15-5x-10}{(x+2)(x-5)} = \frac{-2x-20}{(x+2)(x-5)}.$$

In the method of partial fractions, *we attempt to break down a compound expression* like $\frac{-2x-20}{(x+2)(x-5)}$ into separate fractions of the form

$$\frac{-2x-20}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}.$$

One of the benefit of partial fractions is to evaluate integrals of rational functions. Some forms of integrals involving rational functions are difficult to evaluate directly, which can be made simpler by breaking the fractions into their partial fractions.

20.3.2 How to find partial fractions

The procedure to find partial fractions of a rational function is as follows:

1. If the fraction is improper, perform the algebraic division to make it a proper fractions.
2. Factor the denominator if it is not already in the factored form.

3. If the denominator contains two linear factors, break it into two partial fractions using constant A and B as follows:

$$\frac{-2x-20}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}.$$

4. If the denominator contains some power of a linear factor, we break it down to partial fractions as follows:

$$\frac{-2x-20}{(x+2)^2(x-5)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-5}.$$

5. If the denominator contains a quadratic factor, we break it down to partial fractions as follows:

$$\frac{-2x-20}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}.$$

Example 20.9. Obtain the partial fraction decomposition of $\frac{x-2}{x^2+4x+3}$.

Solution. First of all we factor the denominator as

$$\frac{x-2}{x^2+4x+3} = \frac{x-2}{(x+2)(x+1)}.$$

As the denominator contains two linear factors, we write the partial fractions as

$$\frac{x-2}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}.$$

We simplify right side using common denominator as

$$\frac{x-2}{(x+3)(x+1)} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}.$$

Since the denominators of the left side and the right side are the same, the numerator on the left equal to the numerator on the right, that is,

$$x-2 = A(x+1) + B(x+3). \quad (20.1)$$

Note that, Equation 20.1 is an identity that is true for all values of x . We choose a value of x that will eliminate either A or B on the right. When, $x = -1$,

$$A(x+1) = 0 \implies -1-2 = B(-1+3) \implies B = -\frac{3}{2}.$$

Now, we eliminate B by letting $x = -3$, that is

$$-3-2 = A(-3+1) + B(-3+3) \implies -5 = -2A+0 \implies A = \frac{5}{2}.$$

Therefore,

$$\frac{x-2}{(x+3)(x+1)} = \frac{5/2}{x+3} - \frac{3/2}{x+1}.$$

□

20.3.3 Integration by partial fractions

Example 20.10. Use partial fractions to evaluate $\int \frac{2x-3}{(x-1)^2}$.

Solution. We find the partial fraction first in the form

$$\frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1) + B}{(x-1)^2}.$$

Equating the numerators gives

$$2x-3 = A(x-1) + B \quad (20.2)$$

Setting $x = 1$ in 20.2, will eliminate A , we can find B

$$2(1)-3 = A(1-1) + B \implies B = -1,$$

and setting $x = 2$, will give a relationship between A and B , that is,

$$2(2)-3 = A(2-1) + B \implies A-1 = 1 \implies A = 2.$$

Therefore,

$$\frac{2x-3}{(x-1)^2} = \frac{2}{x-1} - \frac{1}{(x-1)^2}.$$

Now, using the partial fractions and the substitution $x-1 = u$, we have

$$\begin{aligned} \int \frac{2x-3}{(x-1)^2} &= \int \left[\frac{2}{x-1} - \frac{1}{(x-1)^2} \right] dx \\ &= \int \left[\frac{2}{u} - \frac{1}{u^2} \right] du = 2 \ln|u| - \left(-\frac{1}{u} \right) + C \\ &= \ln u^2 + \frac{1}{u} + C = \ln(x-1)^2 + \frac{1}{x-1} + C. \end{aligned}$$

□

Example 20.11. Use the partial fractions to evaluate $\int \frac{x}{x^3-1} dx$.

Solution. We find the partial fractions first in the form

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

On equating the numerators, we get

$$x = A(x^2+x+1) + (Bx+C)(x-1) \quad (20.3)$$

Setting $x = 1$ in 20.3, will eliminate B and C , we can find A , i.e.

$$1 = A(1^2+1+1) + (B(1)+C)(1-1) \implies A = \frac{1}{3}.$$

In order to find B and C , we assign other values to x . By setting $x = 0$ in Equation 20.3, we can eliminate B , as

$$0 = A(0^2 + 0 + 1) + [B(0) + C](0 - 1) \implies A - C = 0 \implies C = A = \frac{1}{3}.$$

Finally, setting $x = 2$ in Equation 20.3 will give a relation between A, B and C , that is,

$$2 = A(2^2 + 2 + 1) + [B(2) + C](2 - 1) \implies 2 = 7A + 2B + C.$$

Therefore,

$$2 = \frac{7}{3} + 2B + \frac{1}{3} \implies B = -\frac{1}{3}$$

Thus,

$$\int \frac{x}{x^3 - 1} dx = \frac{1}{3} dx = \frac{1}{3} \int \frac{1}{x - 1} dx - \int \frac{1}{3} \frac{x - 1}{x^2 + x + 1} dx = \frac{1}{3} I_1 - \frac{1}{3} I_2.$$

Now,

$$I_1 = \frac{1}{x - 1} = \ln|x - 1|.$$

$$\begin{aligned} I_2 &= \int \frac{x - 1}{x^2 + x + 1} dx = \frac{1}{2} \int \frac{2x - 2}{x^2 + x + 1} dx = \frac{1}{2} \int \frac{2x + 1 - 2}{x^2 + x + 1} dx \\ &= \frac{1}{2} \int \frac{2x + 1}{x^2 + x + 1} - \frac{3}{2} \int \frac{dx}{x^2 + x + 1} = \frac{1}{2} I_3 - \frac{3}{2} I_4. \end{aligned}$$

□

Here

$$I_3 = \int \frac{2x + 1}{x^2 + x + 1} = \ln|x^2 + x + 1|,$$

and

$$I_4 = \int \frac{dx}{x^2 + x + 1} = \int \frac{dx}{(x + \frac{1}{2}) + (\frac{\sqrt{3}}{2})^2} \quad (20.4)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x - 3/2}{\sqrt{3}/2} \right) = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right). \quad (20.5)$$

Finally, substituting the values of I_1, I_2, I_3 and I_4 , we arrive at

$$\frac{x}{x^3 - 1} dx = \frac{1}{3} \ln|x - 1| - \frac{1}{3} \left(\frac{1}{2} \ln|x^2 + x + 1| - \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right) \right) + k.$$

Example 20.12. Evaluate $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$.

Solution. The fraction is improper. So, we first perform algebraic division. This enables us to write

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \int \left(x + 2 \frac{4x}{x^3 - x^2 - x + 1} \right) dx = \frac{x^2}{2} + 2x + \int \frac{4x}{x^3 - x^2 - x + 1} dx.$$

Now consider

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{4x}{(x+1)(x-1)^2} dx.$$

So the partial fraction decomposition is

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

This implies

$$4x = (A+C)x^2 + (B-2C)x - (A-B-C).$$

By equating coefficients on the both sides, we get

$$A+C=0 \quad B-2C=0 \quad -A+B+C=0.$$

Solving these simultaneous equations, we get

$$A=1, \quad B=2, \quad C=-1.$$

Therefore,

$$\begin{aligned} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx &= \int \left[x - 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + K \\ &= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln \left| \frac{x-1}{x+1} \right| + K. \end{aligned}$$

□

20.3.4 Algorithm for integrating rational functions

Integration of a rational function $f = \frac{N}{D}$, where N and D are polynomials can be performed as follows.

1. If $\text{degree}(D) \leq \text{degree}(N)$, perform polynomial division and write

$$\frac{N}{D} = S + \frac{R}{D}$$

where S and R are polynomials with $\text{degree}(R) < \text{degree}(D)$.

2. Integrate the polynomial S .
3. Factorize the polynomial D . Perform partial fraction decomposition of $\frac{R}{D}$.
4. Integrate the partial fraction decomposition.